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Total curvature and packing of knots ☆

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Abstract

We establish a new relationship between total curvature of knots and crossing number. If K is a smooth knot in $\mathbb{R}^3$, R the cross-section radius of a uniform tube neighborhood K, L the arclength of K, and κ the total curvature of K, then

$$\text{crossing number of } K < 4\frac{L}{R}\kappa.$$

The proof generalizes to show that for smooth knots in $\mathbb{R}^3$, the crossing number, writhe, Möbius Energy, Normal Energy, and Symmetric Energy are all bounded by the product of total curvature and rope-length.

One can construct knots in which the crossing numbers grow as fast as the $(4/3)$ power of L/R. Our theorem says that such families must have unbounded total curvature: If the total curvature is bounded, then the rate of growth of crossings with ropelength can only be linear.

Our proof relies on fundamental lemmas about the total curvature of curves that are packed in certain ways: If a long smooth curve A with arclength L is contained in a solid ball of radius ρ, then the total curvature of K is at least proportional to L/ρ. If A connects concentric spheres of radii $a \geqslant 2$ and $b \geqslant a+1$, by running from the inner sphere to the outer sphere and back again, then the total curvature of A is at least proportional to $1/\sqrt{a}$.

1. Introduction

The total curvature of smooth closed curve in $\mathbb{R}^3$ must be at least 2π; this is a theorem of Fenchel [7,12]. If the curve actually is a nontrivial knot, then the Fary–Milnor theorem [7,11,12,19] says the total curvature must be $>4\pi$. Are there properties of the knot that could guarantee larger total curvature? Successive composition [13] or other kinds of satellite constructions ([24] together with [19]) will work. On the other hand, topological complexity in the form of high crossing-number is not enough: it is well known at least since [18] that one can construct knots with arbitrarily large minimum crossing-number represented by curves with uniformly bounded total curvature. Here is one way to build them.

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Example 1 *(Knots with bounded total curvature).* Fix any odd integer n. Construct a smooth knot K_n, with minimum crossing number n, as the union of four arcs H, A, C_1, C_2, where the total curvatures are $\kappa(H) \to 0$ as $n \to \infty$, $\kappa(A) = 0$, and $\kappa(C_1) \approx \kappa(C_2) \approx 2\pi$. Let H be the circular helix in $\mathbb{R}^3$ parametrized as $[\cos(t), \sin(t), n^2 t]$, $t = 0, \dots, n\pi$. The height coordinate $n^2 t$ makes $\kappa(H)$ behave like $1/n$ for large n. Using any exponent larger than 1, i.e. $n^{1+\varepsilon} t$, still makes $\kappa(H) \to 0$. Let A be the central axis of the cylinder on which H runs. Let C_1 and C_2 be curves that smoothly connect the top of H to the bottom of A and vice-versa. For large n, the tangent vectors at the beginning and end of H are nearly vertical. The arcs C_1 and C_2 can be chosen to be almost planar-convex curves, with total curvatures $\kappa(C_i) \approx 2\pi$. Similarly, for any (p, q), torus knots or links of type (p meridians, q longitudes), can have total curvature close to $2\pi q$ if they are drawn on a standard torus that is long and thin enough.

In this paper, we show that examples of the preceding kind are, in a sense, the only kind possible. In order to represent an infinite family of knot types with uniformly bounded total curvature, the knots must be "long and thin"; if we imagine them made of actual "rope", then the ratio of length to rope-thickness must grow without bound.

Definition 1. Suppose K is a smooth knot in $\mathbb{R}^3$. For $r > 0$, consider the disks of radius r normal to K, centered at points of K. For r sufficiently small, these disks are pairwise disjoint and combine to form a tubular neighborhood of K. Let $R(K)$, the *thickness radius* of K, denote the supremum of such "good" radii. The *ropelength* of K, denoted $E_L(K)$, is the ratio

$$E_L(K) = \frac{\text{total arclength of } K}{R(K)}.$$

The fundamental properties of thickness radius were developed in [17]. The idea of using the ratio of length-to-radius to measure knot complexity was introduced in [2], and this ratio, denoted $E_L(K)$, is connected to other knot "energies" in [3,4] and [21]. Variations on thickness are developed in [20,8–10,15,16,22,23].

Definition 2. Let K be a smooth knot. From almost every direction, if we project K into a plane, the projection is regular, in particular there are only finitely many crossings. We can average this crossing-number over all directions of projection (i.e. over the almost-all set of directions that give regular projections). This *average crossing number* is denoted $\mathrm{acn}(K)$. Certainly, the minimum crossing-number of the knot-type, $\mathrm{cr}[K]$, satisfies $\mathrm{cr}[K] \leqslant \mathrm{acn}(K)$. We shall rely on the formulation of $\mathrm{acn}(K)$ developed in [14].

Our main result is the following:

Theorem 1. *If K is a smooth knot in $\mathbb{R}^3$, then*

$$\mathrm{acn}(K) < 4E_L(K)\kappa(K).$$

The coefficient $c = 4$ has been rounded up for simplicity. What matters is that the crossing number is essentially bounded by ropelength times curvature. In heuristic discussions, we may omit coefficients altogether.

If we are given some family of knots in which total curvature is uniformly bounded, while crossing number is growing, then the ropelength must be growing at least as fast as the crossing numbers. Alternatively, if the crossing numbers are growing faster than ropelength, then the total curvatures must be growing fast enough to make up the difference. We showed in [3,4] that $\mathrm{acn}(K) \leqslant E_L(K)^{4/3}$, and there are examples [1,5] where the 4/3 power is achieved. In the particular examples of [1,5], the knots and links have evident growing total curvature; our theorem says that some unbounded amount of total curvature must occur in any situation of more-than-linear growth of crossings with ropelength.

If we model a knot made of actual "rope" as a smooth curve with a uniform tube neighborhood, then the thickness (radius) r of that rope is $\leqslant R(K)$, so $E_L(K) \leqslant \frac{L}{r}$. Thus the theorem also holds with $\frac{L}{r}$ in place of E_L.

2. Lemmas on total curvature

The three lemmas in this section establish fundamental properties of smooth space-curves, relating total curvature to packing, to oscillation relative to a given point, and to the "illumination" of a given point. We deal in this section with smooth space-curves, not assuming the curves are simple or closed; and we make no use of thickness. Also we do not assume the curves have finite length.

To keep the arguments as simple as possible, we assume throughout the paper that "smooth" means smooth of class C^2. The lemmas and theorem can be adapted for curves that are piecewise smooth. For a smooth curve A, we denote the total curvature of A by $\kappa(A)$.

It is intuitively clear that if a long rope is packed in a small box, then the rope must curve a lot. This fundamental lemma is an important ingredient in our analysis of the interplay between ropelength, crossing number, and total curvature.

A ball of radius ρ contains, of course, a diameter of length 2ρ. But once we postulate length $> 2\rho$, an arc in the ball must curve. In this version, we use $L \geqslant 3\rho$, but any constant larger than 2 will produce some guaranteed amount of total curvature. Inequality (1) and the proof below are taken from [6], with a slight adjustment for non-closed curves. If the curve is closed, then the number 2 can be omitted from (1). We learned of this proof from lecture notes of S. Tabachnikov, now available as [25].

Lemma 2.1 *(Packing and curvature). Suppose A is a smooth connected curve of length L, contained in a round 3-ball of radius ρ. Then $\kappa(A)$ is approximately proportional to at least L/ρ. More precisely, letting κ denote $\kappa(A)$, we have the following,*

$$L \leqslant \rho(\kappa + 2), \tag{1}$$

which gives

$$L \geqslant 3\rho \implies \kappa \geqslant 1. \tag{2}$$

Proof. Translate the ball and curve so the center of the ball is at the origin. Let $s \to x(s)$, $s \in [0, L]$, be a unit speed parametrization of A. Since $|x'(s)| = 1$, we can write

$$L = \int_{s=0}^{L} x'(s) \cdot x'(s).$$

Integrate by parts to get

$$L = x'(s) \cdot x(s)\big|_0^L - \int_0^L x(s) \cdot x''(s).$$

Since $|x'(s)| = 1$, and $|x(s)| \leqslant \rho$, the first term is at most 2ρ and the second term is at most $\rho\kappa$. $\square$

Another basic way that a long curve is forced to have a guaranteed amount of total curvature is if its distance from some given point oscillates. This is captured in the next lemma.

Lemma 2.2 *(Oscillation and curvature). Let S_a, S_b be concentric spheres with radii $a < b$. Let A be a smooth curve that starts at a point of S_a, somewhere touches the sphere S_b, and ends at a point of S_a. Then the total curvature is at least approximately on the order of $1/\sqrt{a}$.*

More precisely,

$$\kappa(A) \geqslant \pi - 2\arcsin(a/b). \tag{3}$$

If $b \geqslant a + 1$, then

$$\kappa(A) > \frac{2\sqrt{2}}{\sqrt{a+1}}. \tag{4}$$

If $a \geqslant 2$, the bound (4)*, along with simplifying the coefficient, gives*

$$\kappa(A) > \frac{2}{\sqrt{a}}.$$

Remark. One can think of the arc in this lemma as being just contained in the spherical shell bounded by the two given spheres, or as reaching out past S_b, just so it returns back to end on S_a.

Proof. We rely on [19] to reduce the proof of (3) to analyzing a certain triangle, and then calculate (4).

Let p, q be the endpoints of A, z a point of $A \cap S_b$, $\overline{pz}$ and $\overline{zq}$ the line segments from p to z and from z to q, and let P be the two-edge polygon $\overline{pz} \cup \overline{zq}$. Since the polygon P is an inscribed polygon of A, we know from [19] that $\kappa(A) \geqslant \kappa(P)$, so it suffices to establish the desired lower bound for $\kappa(P)$. If either of the edges of P is not tangent to the sphere S_a, we can pivot the edge at point z to move the edge to tangency in a way that opens the angle $\widehat{pzq}$, so reducing $\kappa(P)$. Thus, it suffices to prove the lower bound for tangent two-edge polygons.

The three points p, z, q determine a plane; we wish that plane also would include the center of the spheres. If not, then (keeping the two edges tangent to S_a, and allowing the points of tangency and the angle $\widehat{pzq}$ to change), rotate the plane of p, z, q (with axis of rotation the line through z parallel to $\overline{pq}$) until it does contain the center of the spheres. This deformation also would increase the angle $\widehat{pzq}$, and so decrease the total curvature of P. Thus, we are reduced to the situation where p, z, q and the center of the spheres are coplanar, and P consists of tangent lines to S_a drawn symmetrically from point z on S_b. We then have a right-triangle with

$$\sin\left(\frac{\widehat{pzq}}{2}\right) = \frac{a}{b},$$

which gives (3).

To derive (4), first note that $\pi - 2\arcsin(a/b)$ increases as $(b-a)$ gets larger. So if we show $\pi - 2\arcsin(a/b) \geqslant \frac{2\sqrt{2}}{\sqrt{a+1}}$ for $b = a+1$, then we will have that inequality for all $b \geqslant a+1$.

Rewrite (3) in terms of a and $b = a+1$,

$$\kappa(A) \geqslant \pi - 2\arcsin\left(1 - \frac{1}{a+1}\right).$$

Now let $t = \sqrt{\frac{1}{a+1}}$ and check (analytically or graphically) that

$$\pi - 2\arcsin\left(1 - t^2\right) > (2\sqrt{2})t. \qquad \square$$

In the next lemma, we call the curve Y instead of A, to help clarify how the lemmas will be used later: We will prove Lemma 2.3 by applying Lemmas 2.1 and 2.2 to subarcs A of Y.

Suppose Y is a smooth curve in $\mathbb{R}^3$, and x_0 is a point some finite distance from Y. The integral

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} \tag{5}$$

can be thought of as measuring the "illumination" of x_0 by Y.

Lemma 2.3 *(Illumination and curvature). Suppose Y is a smooth curve in $\mathbb{R}^3$, and x_0 is a point such that $\forall y \in Y$, $|y - x_0| \geqslant 2$. Then the illumination of x_0 by Y is bounded by the total curvature of Y. More precisely,*

$$\int_{y \in Y} \frac{1}{|y - x_0|^2} \leqslant c_1 + c_2\kappa(Y), \tag{6}$$

where c_1, c_2 are universal constants independent of Y (values $c_1 = 16$ and $c_2 = 43$ are sufficient).

Lemma 2.3 is perhaps the most intricate part of the paper. Before proving it, we present four special cases. The general argument does not reduce to these special cases—rather we include them to give an intuitive sense of why the proposition might be true (the first four), and some of the issues one needs to confront in building a proof (the fifth).

2.1. Special cases for Lemma 2.3

2.1.1. A spiral to show the lemma is sharp in the power of $\kappa(Y)$

Let Y be the polar coordinates curve $r = 3 - 1/\theta$, $\theta = 1, \ldots, \Theta$. As Θ increases, the illumination (of $x_0 =$ the origin) is asymptotic to $\frac{1}{3}\kappa(Y)$.

2.1.2. Y is a ray

Suppose Y is a straight line, starting at a point 2 units from x_0 and aiming radially away from x_0. Then the line integral is just $\int_2^\infty 1/s^2\,\mathrm{d}s = 1/2$.

2.1.3. Y is a straight line

Suppose Y is a straight line, infinite in both directions, and tangent to the sphere of radius 2 centered at x_0. Then

$$\int\limits_{y\in Y} \frac{1}{|y-x_0|^2} = \int\limits_{-\infty}^{\infty} \frac{1}{4+s^2}\,\mathrm{d}s = \frac{\pi}{2}.$$

If the line is a finite segment, or the minimum distance from Y to x_0 is >2, then the integral is $<\pi/2$.

2.1.4. Y is a certain kind of polygon

Suppose Y' is a polygonal path (or closed curve) consisting of e edges (of possibly varying lengths), such that each pair of consecutive edges meets at a right angle. Form a smooth curve Y by replacing the corners of Y' with small quarter-circles. Then, by the second special case, each edge of Y contributes $<\pi/2$ to the illumination integral, so

$$\int\limits_{y\in Y} \frac{1}{|y-x_0|^2} < e\frac{\pi}{2} = \begin{cases} \kappa(Y) & \text{if the polygon has endpoints, or} \\ \kappa(Y)+\pi/2 & \text{if the polygon is closed.} \end{cases}$$

2.1.5. Y is a monotone arc

Suppose Y is a smooth curve, starting at a point y_0 with $|y_0 - x_0| = 2$, with the property that the distance function $|y - x_0|$ is monotone increasing on Y.

For $n = 2, 3, \ldots,$ let $B[n]$ denote the round ball of radius n centered at x_0, and let $S[n, n+1]$ denote the spherical shell with radii n and $n+1$. By our assumption of monotonicity, each intersection $Y \cap S[n, n+1]$ is a connected arc, which we denote Y_n. Then

$$\int\limits_{y\in Y} \frac{1}{|y-x_0|^2} = \sum_{n=2}^{\infty} \int\limits_{y\in Y_n} \frac{1}{|y-x_0|^2} \leqslant \sum_{n=2}^{\infty} \frac{\ell(Y\cap S[n,n+1])}{n^2}.$$

We would like to bound each of the numbers $\ell(Y \cap S[n, n+1])$, in terms of total curvature of Y, somehow using Lemma 2.1. That lemma gives upper bounds for the lengths $\ell(Y \cap B[n])$ in terms of total curvature, but does not explicitly bound the amounts in given shells. We get around this problem by bounding (not the illumination integral from Y itself, but rather) the illumination integral for a hypothetical curve Y^* that is packed around x_0 in such a way as to make the illumination integral as large as possible subject to the constraints imposed by Lemma 2.1. (In this intuitive discussion of the monotone case, we will continue with the image of a "hypothetical curve". In the actual proof of Lemma 2.3, we will be more rigorous.)

For brevity, let κ denote $\kappa(Y)$. Since $Y \cap \operatorname{int} B[2] = \emptyset$, we start with $Y_2 = Y \cap B[3]$. By inequality (1),

$$\ell\big(Y \cap B[3]\big) \leqslant 3(\kappa + 2).$$

Similarly,

$$\ell\big(Y \cap B[4]\big) \leqslant 4(\kappa + 2),$$
$$\ell\big(Y \cap B[5]\big) \leqslant 5(\kappa + 2), \quad \text{etc.}$$

If the curve Y does not actually achieve these bounds, then add extra length (the "hypothetical" curve Y^*) in each of the shells as needed to actually reach these bounds. Since we are adding length to the Y that already exists, the illumination integral can only increase. Thus the illumination for Y^* is an upper bound for the illumination for Y.

We have

$$\begin{aligned}
&\ell\big(Y^* \cap B[3]\big) = 3(\kappa+2),\\
&\ell\big(Y^* \cap B[4]\big) = 4(\kappa+2),\\
&\ell\big(Y^* \cap B[5]\big) = 5(\kappa+2), \quad \text{etc.}
\end{aligned}$$

Thus

$$\begin{aligned}
&\ell\big(Y^* \cap S[2,3]\big) = 3(\kappa+2),\\
&\ell\big(Y^* \cap S[3,4]\big) = 4(\kappa+2) - 3(\kappa+2) = (\kappa+2),\\
&\ell\big(Y^* \cap S[4,5]\big) = 5(\kappa+2) - 4(\kappa+2) = (\kappa+2), \quad \text{etc.}
\end{aligned}$$

And so,

$$\int\limits_{y\in Y} \frac{1}{|y-x_0|^2} \leqslant \int\limits_{y\in Y^*} \frac{1}{|y-x_0|^2} < \frac{3(\kappa+2)}{2^2} + \sum_{n=3}^{\infty} \frac{(\kappa+2)}{n^2} < 2\kappa+3.$$

2.2. *Proof of Lemma 2.3*

We begin as we did in Section 2.1.5. For $n = 2, 3, \ldots$, let $B[n]$ denote the round ball of radius n centered at x_0, and let $S[n, n+1]$ denote the spherical shell with radii n and $n+1$. We need to bound the total arclength of Y contained in each shell $S[n, n+1]$, but we cannot do this directly since we are not assuming monotonicity as in Section 2.1.5. The shell-intersections might consist of long arcs, or might consist of unions of many short arcs, as Y meanders in space, close to, or far from, x_0.

When Y is contributing to the integral by having long arcs close to x_0, we can infer curvature from Lemma 2.1. If Y is contributing to the integral by oscillating in and out from x_0, we can use Lemma 2.2 to infer curvature.

To implement this plan, and handle the problem of very small oscillations, we are going to translate the problem into discrete combinatorics.

2.2.1. *Assume finite length*

If Y has infinite length, express Y as an increasing union of curves of finite length. Since the constants c_1, c_2 do not depend on the curve Y, we can apply inequality (6) to each of these and observe that both sides of inequality (6) converge appropriately.

2.2.2. *Cut Y into small pieces*

Pick any integer $M > \ell(Y)$, and cut Y into M consecutive arcs Y_i of equal length. Let ε denote the length of each subarc, and note $\varepsilon < 1$.

We assign to each arc Y_i a label $1, 2, 3, \ldots$ representing the shell $S[n, n+1]$ that (perhaps only approximately) contains Y_i. Specifically, if $Y_i \subset S[n, n+1]$, assign label n. If Y_i is not entirely contained in one shell, then it must intersect a sphere $S[n]$; because $\varepsilon < 1$, Y_i can intersect at most one sphere $S[n]$; we assign that label n to the arc. Note that if an arc Y_i carries label n, then $Y_i \subset S(n-1, n+1]$ and $Y_i \cap S[n, n+1] \neq \emptyset$. The set of possible labels is $\{2, \ldots, M+1\}$.

2.2.3. *Discretize the problem*

For each integer n, let $\phi(n)$ denote the total number of arcs Y_i that are labeled n. Thus

$$\int\limits_{y\in Y} \frac{1}{|y-x_0|^2} = \sum_i \int\limits_{y\in Y_i} \frac{1}{|y-x_0|^2} < \sum_{n\geqslant 2} \phi(n) \frac{\varepsilon}{(n-1)^2}. \tag{7}$$

We also need the auxiliary function that counts the total number of arcs that are labeled between 2 and n. Define

$$\Phi(n)=\sum_{j=2}^{n}\phi(j).$$

We proceed as follows:

(1) Abstract the arc Y as the string of integers $\mathcal{L}_Y=\langle a_1,a_2,\dots,a_M\rangle$, where a_i is the shell label of Y_i.
(2) Show $\mathcal{L}_Y$ is constrained in certain ways.
(3) Find a bound for $\Phi(n)$, in terms of $\kappa(Y)$, using Lemmas 2.1 and 2.2.
(4) Note that the functions ϕ and Φ make sense for any finite string $\mathcal{L}$ of integers.
(5) For any finite string $\mathcal{L}$ of integers $\geqslant 2$, define an "energy"

$$E(\mathcal{L})=\sum_{n\geqslant 2}\phi(n)\frac{\varepsilon}{(n-1)^2}.$$

(6) Construct a string $\mathcal{L}^*$ of integers $\in\{2,\dots,M+1\}$ for which we know bounds on the numbers $\phi(n)$, and for which we know $E(\mathcal{L})\leqslant E(\mathcal{L}^*)$.
(7) Find a bound for the value $E(\mathcal{L}^*)$, which is an upper bound for the final sum in (7), of the form we want.

Let $\mathcal{L}_Y$ be the string of shell labels associated to Y. In order to bound $\Phi(n)$, we first establish certain properties of $\mathcal{L}_Y$.

2.2.4. Constraints on $\mathcal{L}_Y$

To shorten formulas in the rest of the proof of Lemma 2.3, we will use κ to denote $\kappa(Y)$.

Since the arcs Y_i are listed in their order along Y, each intersection $Y_i\cap Y_{i+1}$ is nonempty. A substring such as $\langle 34434543\rangle$ is possible. On the other hand, there cannot be a substring such as $\langle 34435543\rangle$, because subarcs with labels 3 and 5 are contained in the disjoint half-open shells $S(2,4]$ and $S(4,6]$. So the first constraint is:

- $\mathcal{L}_Y$ must consist of contiguous labels.

For a given value of n, there cannot be too many long substrings of $\mathcal{L}_Y$ consisting of labels $\leqslant n$. A substring of $\mathcal{L}_Y$ consisting of q symbols $\leqslant n$ represents a connected arc $A\subset Y$ of length $q\varepsilon$ contained in the ball $B[n+1]$. If q is such that $q\varepsilon\geqslant 3(n+1)$, then, by Lemma 2.1(2), $\kappa(A)\geqslant 1$. Thus, for such q, we can have no more than κ such strings. So the second constraint is:

- For each n, the string $\mathcal{L}_Y$ contains at most κ pairwise disjoint substrings of length $\frac{3(n+1)}{\varepsilon}$ consisting of integers $\leqslant n$.

In Section 2.2.5, we will apply this formula to substrings of $\mathcal{L}_\mathcal{Y}$ with entries up through $(n+1)$. We also want to phrase the bound in terms of what $\mathcal{L}_Y$ *cannot* contain. Specifically,

- For each n, the string $\mathcal{L}_Y$ cannot contain $(\kappa+1)$ pairwise disjoint substrings of length $\frac{3(n+2)}{\varepsilon}$ consisting of integers $\leqslant(n+1)$.

We obtain a third constraint, this time on "jumps". If, in the string $\mathcal{L}_Y$, we observe a substring $\langle n\dots n+1\dots n\rangle$, we cannot infer any particular contribution to total curvature that is independent of ε. But if we see $\langle n\dots n+2\dots n\rangle$, then we can. Let us call a substring $\lambda=\langle n\dots n+2\dots n\rangle$ of $\mathcal{L}_Y$ a *jump at level n*. Two jumps are *non-overlapping* if they are disjoint, or meet in at most one term a_i (of necessity, then, an endpoint of each).

An arc Y_i with label n has nonempty intersection with $S[n,n+1]$; an arc with label $n+2$ intersects $S[n+2,n+3]$. Thus if λ is a jump at level n, then the subarc of Y determined by λ has a subarc A that starts at the sphere $S[n+1]$ and reaches as far out as some $S[b]$, $b\geqslant n+2$, before heading back to end at $S[n+1]$. By Lemma 2.2, such an arc contributes more than $\frac{2}{\sqrt{n+1}}$ to total curvature. Thus we have our third constraint:

- For each n, the string $\mathcal{L}_Y$ cannot have $\frac{1}{2}\kappa\sqrt{n+1}$ non-overlapping jumps of level n.

2.2.5. Bound $\Phi(n)$

We now combine the constraints on substrings and jumps in $\mathcal{L}_Y$ to get bounds on $\Phi(n)$.

Proposition 2.1. *For the string $\mathcal{L}_Y$, for each $n \geqslant 2$,*

$$\Phi(n) < 8\kappa\frac{n^{3/2}}{\varepsilon} + 6\frac{n}{\varepsilon}. \tag{8}$$

Proof. Suppose, to the contrary, that for some n, $\mathcal{L}_Y$ does have that many symbols $2, 3, \ldots, n$. The bound (8) was chosen to be a simple expression that, for $n \geqslant 2$, dominates

$$(\kappa+1)\left(\frac{3(n+2)}{\varepsilon}\right) + \left(\frac{1}{2}\kappa\sqrt{n+1}\right)\left(\frac{3(n+2)}{\varepsilon}\right).$$

Visualize the string $\mathcal{L}_{\mathcal{Y}}$ so that, temporarily, only the symbols $2, 3, \ldots, n$ are visible. Parse these into pairwise disjoint substrings of length $\frac{3(n+2)}{\varepsilon}$. By assumption, we have (many) more than $\kappa + 1$ of these substrings. So in the actual string $\mathcal{L}_Y$, a number of these substrings must get broken up by inserted symbols $\geqslant n+1$. Now make all the symbols $a_i = n+1$ in $\mathcal{L}_Y$ visible as well. These certainly can break up substrings consisting only of symbols $2, 3, \ldots, n$, but they offer no improvement on our situation of exceeding the total curvature bound: we chose the lengths of the substrings to be large enough that even if they were made from symbols $2, 3, \ldots, n+1$, they would still each be contributing $\geqslant 1$ to total curvature, so we cannot have more than κ of these. Thus we must have some symbols $\geqslant n+2$ in the original string $\mathcal{L}_Y$ to break up a number of the "offending" substrings. How many of the substrings can be broken by inserting symbols $a_j \geqslant n+2$? Each offending substring that gets broken this way represents at least one jump at level n. So we must have fewer than $\frac{1}{2}\kappa\sqrt{n+1}$ such interruptions. We are assuming $\Phi(n)$ is large enough that the number of offending substrings is greater than the number of possible interruptions plus the maximum number we could tolerate to be uninterrupted. We conclude that $\Phi(n)$ cannot be that large. □

2.2.6. Construct $\mathcal{L}^*$

We have completed step 3 of our plan, and now proceed. The functions ϕ and Φ make sense for abstract finite strings $\mathcal{L}$ of integers:

$$\phi(n) = \text{number of symbols } a_i \text{ of } \mathcal{L} \text{ that are } n;$$

$$\Phi(n) = \sum_{j=2}^{n} \phi(j).$$

And we can define the "energy"

$$E(\mathcal{L}) = \sum_{n\geqslant 2} \phi(n)\frac{\varepsilon}{(n-1)^2}.$$

We want to construct a string $\mathcal{L}^*$ whose energy we can bound, but also whose energy is larger than $E(\mathcal{L}_Y)$. We do this by successive modification of $\mathcal{L}_Y$. We will denote the new strings $\mathcal{L}_2, \mathcal{L}_3, \ldots$, and denote the corresponding functions $\phi_2, \Phi_2, \phi_3, \Phi_3$, etc. Let us also introduce notation for the bounds in inequality (8):

$$\beta(n) = 8\kappa\frac{n^{3/2}}{\varepsilon} + 6\frac{n}{\varepsilon}.$$

The strings $\mathcal{L}_m$ will have the following properties:

- $\Phi_m(n) \leqslant \beta(n)$ for all n.
- $\Phi_m(n) = \beta(n)$ for $n = 2, \ldots, m$.
- $E(\mathcal{L}_Y) \leqslant E(\mathcal{L}_m)$.

If we take a string $\mathcal{L}$ and change some symbol a_i to a lower integer, that increases $E(\mathcal{L})$. Also, if we introduce a new additional symbol a_j somewhere in $\mathcal{L}$, that increases $E(\mathcal{L})$.

We begin the construction by adjoining to $\mathcal{L}_Y$ enough terms $a_j = (M+1)$ to raise $\Phi(M+1)$ to equal $\beta(M+1)$. Call this string $\mathcal{L}_1$. All the other strings $\mathcal{L}_m$ will be obtained by changing various terms of $\mathcal{L}_1$ to lower values, thus raising energy while keeping $\Phi(M+1)$ unchanged.

We know in $\mathcal{L}_Y$ that $\phi(2) = \Phi(2) < 8\kappa\frac{2^{3/2}}{\varepsilon} + 6\frac{2}{\varepsilon}$. To construct $\mathcal{L}_2$, change enough 3's in $\mathcal{L}_Y$ to 2 to bring the number of 2's up to $\beta(2)$. [Note: For simplicity, we will use the bound itself, rather than rounding up if it is not an integer.] Changing a 3 to a 2 has no effect on $\Phi(n)$ for $n \geqslant 3$. If there are not enough 3's (i.e. if $\Phi(3) < \beta(2)$), we change 4's to 2's. This increases $\Phi(3)$ up to $\beta(2)$, still $< \beta(3)$; and values $\Phi(n)$ are unchanged for $n \geqslant 4$. If there are not enough 4's, we change 5's, etc. Continuing in this way, we obtain a string $\mathcal{L}_2$ with the properties

(1) $\phi_2(2) = \Phi_2(2) = \beta(2)$,
(2) $\Phi_2(n) \leqslant \beta(n)$ for $n \geqslant 3$, and
(3) $E(\mathcal{L}_2) \geqslant E(\mathcal{L}_Y)$.

We next want to make $\phi_2(3)$ large enough that $\Phi_2(3) = \beta(3)$. So we again change higher labels, first change 4's to 3's, then (if necessary) 5's to 3's, etc.

We continue inductively to construct $\mathcal{L}_3, \ldots, \mathcal{L}_{(M+1)} = \mathcal{L}^*$, so that

(1) For $n = 2, \ldots, m$, $\Phi_m(n) = \beta(n)$.
(2) For $n = (m+1), \ldots, (M+1)$, $\Phi_m(n) \leqslant \beta(n)$.
(3) $E(\mathcal{L}_m) \geqslant E(\mathcal{L}_Y)$.

Because we know exactly the values $\Phi^*(n)$, we can compute the values $\phi^*(n)$ and so bound the energy.

$$\phi^*(2) = \beta(2) = 8\kappa\frac{2^{3/2}}{\varepsilon} + 6\frac{2}{\varepsilon},$$

and for $n \geqslant 3$,

$$\phi^*(n) = \Phi^*(n) - \Phi^*(n-1) = \beta(n) - \beta(n-1) = 8\kappa\frac{n^{3/2} - (n-1)^{3/2}}{\varepsilon} + \frac{6}{\varepsilon}.$$

2.2.7. Bound $E(\mathcal{L}^*)$

We bound the energy of $\mathcal{L}^*$ by passing to an infinite sum, so the value of M is immaterial. Since we know the values $\phi^*(n)$, we have

$$\begin{aligned} E(\mathcal{L}^*) &= \sum_{n=2}^{M+1} \phi^*(n)\frac{\varepsilon}{(n-1)^2} \\ &< 8\kappa 2^{3/2} + 12 + 8\kappa\sum_{n=3}^{\infty}\frac{n^{3/2} - (n-1)^{3/2}}{(n-1)^2} + 6\sum_{n=3}^{\infty}\frac{1}{(n-1)^2} \\ &< 16 + 43\kappa. \end{aligned}$$

3. Proof of Theorem 1

As a preliminary step, rescale the knot so the thickness radius $R(K) = 1$. This has no effect on the total curvature or on the average crossing number, and simplifies the ratio $E_L(K)$ to just the length, L. We want to show

$$\operatorname{acn}(K) \leqslant c \cdot L \cdot \kappa(K),$$

where c is some coefficient that works for all knots.

The average crossing number of a knot can be expressed as an integral over the knot [14], similar to Gauss's double integral formula for the linking number of two loops. Specifically,

$$\mathrm{acn}(K) = \frac{1}{4\pi} \int\limits_{x\in K} \int\limits_{y\in K} \frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3},$$

where T_x, T_y are the unit tangents at x, y and $\langle u, v, w\rangle$ is the triple scalar product $(u \times v)\cdot w$ of the three vectors u, v, w.

Write the double integral as a sum of two terms:

$$\mathrm{Near}(K) = \int\limits_{x\in K} \int\limits_{\mathrm{arc}(x,y)\leqslant\pi} \frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3},$$

and

$$\mathrm{Far}(K) = \int\limits_{x\in K} \int\limits_{\mathrm{arc}(x,y)\geqslant\pi} \frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3}.$$

We shall analyze these contributions separately in the next two sections, and find bounds of the form

$$\mathrm{Near}(K) \leqslant b_1 L,$$
$$\mathrm{Far}(K) \leqslant c_1 L + c_2 L\kappa(K),$$

where the coefficients are independent of K. In each case, we bound the inner integral and then multiply by L to bound the double integral.

Combining Near and Far, we get a bound for any smooth curve K of the form

$$\mathrm{acn}(K) \leqslant aL + bL\kappa(K).$$

But if K is a *closed* curve, then by Fenchel's theorem, $\kappa(K) \geqslant 2\pi$. Thus letting $c = \frac{1}{4\pi}(b + \frac{a}{2\pi})$, we have

$$\mathrm{acn}(K) \leqslant cL\kappa(K).$$

Using the values of c_1 and c_2 from Lemma 2.3 and b_1 from Section 3.1, we get $c \approx 3.8$.

3.1. *Bounding* Near(K)

We shall show that the inner integral is uniformly bounded, independent of K.

For any smooth curve with thickness radius R, it is shown in [17] that the curvature at each point is at most $1/R$. So in the present situation, we know that the curvature of K is everywhere $\leqslant 1$.

Let $\theta \to x(\theta)$ be a unit speed parametrization of K. So $x'(\theta) = T_x$ and $|x''(\theta)| \leqslant 1$. We are studying points y for which $\mathrm{arc}(x, y) \leqslant \pi$, so we can take for the parameter set the interval $[0, \pi]$, with our starting point $x = x(0)$ and $y = y(\theta)$ for some $\theta \in [0, \pi]$. Using the same parameter set, let $\theta \to p(\theta)$ be an arclength preserving parametrization of the unit semi-circle. Since the curvature of K is everywhere bounded by the curvature of the unit circle, Schur's theorem [7] tells us that for each θ,

$$\bigl|x(\theta) - x(0)\bigr| \geqslant \bigl|p(\theta) - p(0)\bigr|,$$

that is,

$$|y - x| \geqslant \sqrt{2 - 2\cos\theta}.$$

Thus

$$\frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3} \leqslant \frac{|\langle T_x, T_y, \frac{x-y}{|x-y|}\rangle|}{2-2\cos\theta} = \frac{|\langle T_x, T_y, \frac{x-y}{\theta}\rangle|}{2-2\cos\theta}\,\frac{\theta}{|x-y|}.$$

Using Schur's theorem again, we have $|x-y| \geqslant |p(\theta)-p(0)| = \sqrt{2-2\cos\theta}$. The function $\frac{\theta}{\sqrt{2-2\cos\theta}}$ is increasing on $[0,\pi]$, with maximum value $\pi/2$. So

$$\frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3} \leqslant \frac{\pi}{2}\frac{|\langle T_x, T_y, \frac{x-y}{\theta}\rangle|}{2-2\cos\theta}.$$

The vectors T_y and $\frac{x-y}{\theta}$ are each first-order (in terms of θ) close to T_x. Specifically, we have for T_y,

$$T_y = T_x + \int_{t=0}^{\theta} x''(t).$$

Since $|x''| \leqslant 1$, this says we can write T_y as $T_x + V$, where $|V| \leqslant \theta$.

On the other hand, the fundamental theorem of calculus, applied first to $x(\theta)$ and then again to $x'(s)$, says

$$y = x(\theta) = x(0) + \int_{s=0}^{\theta} x'(s)\,\mathrm{d}s = x(0) + \theta x'(0) + \int_{s=0}^{\theta}\int_{u=0}^{s} x''(u)\,\mathrm{d}u\,\mathrm{d}s,$$

so we can write $\frac{|x-y|}{\theta}$ as $T_x + W$, where $|W| \leqslant \frac{1}{2}\theta$.

We now have $T_x \times T_y = T_x \times (T_x + V) = T_x \times V$, which is a vector perpendicular to T_x with length $\leqslant\theta$. When we take the dot product of this vector with $T_x + W$, we just get the dot product with W, so a number whose size is at most $\frac{1}{2}\theta^2$.

We now have

$$\frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3} \leqslant \frac{\pi}{4}\frac{\theta^2}{2-2\cos\theta} \leqslant \frac{\pi}{4}\left(\frac{\pi}{2}\right)^2,$$

so the inner integral is bounded by $b_1 = (2\pi)\cdot(\frac{\pi}{4})\cdot(\frac{\pi}{2})^2$, since the points y run from (what we might denote as) $x-\pi$ to $x+\pi$.

Multiply this bound for the inner integral by L to bound the double integral.

3.2. *Bounding* Far(K)

As in the previous case, we shall bound the inner integral,

$$\int_{\operatorname{arc}(x,y)\geqslant\pi} \frac{|\langle T_x, T_y, x-y\rangle|}{|x-y|^3},$$

then multiply by L to bound the double integral.

As before, we write the integrand as the triple scalar product of three unit vectors, divided by $|x-y|^2$. Since the numerator has magnitude at most 1, it suffices to bound

$$\int_{\operatorname{arc}(x,y)\geqslant\pi} \frac{1}{|x-y|^2}.$$

For any smooth curve with thickness radius R, it is shown in [17] that points x, y with $\operatorname{arc}(x,y) \geqslant \pi R$ must have $|x-y| \geqslant 2R$. So in our situation, when $\operatorname{arc}(x,y)\geqslant\pi$, we know $|x-y|\geqslant 2$.

Fix x and let $Y = \{y \in K \mid \operatorname{arc}(x,y) \geqslant \pi\}$. By Lemma 2.3,

$$\int_Y \frac{1}{|y-x|^2} \leqslant c_1 + c_2\kappa(Y) \leqslant c_1 + c_2\kappa(X).$$

Thus

$$\operatorname{Far}(K) \leqslant c_1 L + c_2\kappa(X)L.$$

4. Knot energies

The analysis of knot energies $E_N(K)$ and $E_S(K)$ in [3,4] and the Möbius energy E_O in [21] are similar to the analysis of average crossing number here: All involve bounding "Near" and "Far" contributions, and all rely on bounding $\int_y \frac{1}{|y-x|^2}$ for the "Far" part. We can use Lemma 2.3 to show that each of these energies is bounded by [some constant, that depends on the energy but not on K, times] $E_L(K)\kappa(K)$.

5. Can the theorem be improved?

Our theorem says (throughout this section, we will suppress coefficients)

$$\operatorname{acn}(K) \leqslant E_L(K)\kappa(K).$$

Is it possible to lower the exponent (from 1) on one or both of $E_L(K)$, $\kappa(K)$? As noted in Example 2.1.1, Lemma 2.3 is sharp. However, if we include thickness, and postulate that the knot is long and distributed homogeneously in space, then we can argue heuristically that there would be a lower-order bound. This leads to the conjecture that in fact $\operatorname{acn}(K) \leqslant E_L(K)\kappa(K)^{1/2}$.

Scale the knot K so it has thickness radius $r(K) = 1$. Then $E_L(K)$ is just the arclength, L, of K. Suppose K is distributed in space so that relative to each point $x_0 \in K$, each spherical shell $S[n, n+1]$ about x_0 contains on the order of n^β arclength of K. Here β is constant, independent of the choice of x_0, and is a measure of the density of packing of K. Fix some $x_0 \in K$. The shells run from $n = 0$ to whatever value N (for that x_0) is needed to engulf all of K.

The amount of arclength of K in each shell has to include at least enough to reach from one sphere to the other, a constant, so $\beta \geqslant 0$. On the other hand, since $r(K) = 1$, an arc (or union of arcs) of K of some total length ℓ carries with it a proportional amount of excluded volume $= \pi\ell$. Since the volume of a spherical shell is approximately proportional to the area of a boundary sphere, we must have $\beta \leqslant 2$.

The total arclength L of K is the sum of the amounts in the shells, so if the amount in each shell is on the order of n^β, then $L \approx N^{\beta+1}$.

Assuming K is long enough, relative to N, to apply Lemma 2.1, we have $\kappa(K) \geqslant L/N \approx N^{\beta+1}/N = N^\beta$.

We proceed as in the proof of the main theorem: $\operatorname{Near}(K)$ is bounded by a constant and we will bound $\operatorname{Far}(K)$ by bounding the inner Illumination integral and multiplying by L.

If the amount of length of K in each shell $S[n, n+1]$ is on the order of n^β, then

$$\text{Illumination} \leqslant \sum_{n=2}^{N} \frac{n^\beta}{n^2} \approx \begin{cases} \text{constant} - N^{\beta-1} & (0 \leqslant \beta < 1), \\ \log N & (\beta = 1), \\ N^{\beta-1} & (1 < \beta \leqslant 2). \end{cases}$$

In the first situation ($0 \leqslant \beta < 1$), we get a bound on average crossing number proportional to L, that is $\operatorname{acn}(K) \leqslant a_1 + a_2 E_L(K)$. This situation includes long, thin knots, as well as knots such as iterated composites of congruent curves where crossing number and ropelength grow at the same rates. In the second situation ($\beta = 1$), we have $\operatorname{acn}(K) \leqslant a_1 + a_2 E_L(K) \log \kappa(K)$. In the third situation ($1 < \beta \leqslant 2$), we have

$$\operatorname{acn}(K) \leqslant a_1 + a_2 E_L(K)\kappa(K)^{\frac{\beta-1}{\beta}}.$$

When $\beta = 2$, we have the densest possible spatial packing of K as in the examples [1,5], where the growth rate $\operatorname{acn}(K) \approx E_L(K)\kappa(K)^{1/2}$ is attained.

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